Chapter 22

Parametric Equations

Imagine a car is traveling along the highway and you look down at the situation from high above:

![Diagram of a car on a highway with a dynamic viewpoint.]

**Figure 22.1:** The dynamic motion of a car on a static highway.

We can adopt at least two different viewpoints: We can focus on the entire highway all at once, which is modeled by a curve in the plane; this is a “static viewpoint”. We could study the movement of the car along the highway, which is modeled by a point moving along the curve; this is a “dynamic viewpoint”. The ideas in this chapter are “dynamic”, involving motion along a curve in the plane; in contrast, our previous work has tended to involve the “static” study of a curve in the plane. We will combine our understanding of linear functions, quadratic functions and circular functions to explore a variety of dynamic problems.
22.1 Parametric Equations

Example 22.1.1. After a vigorous soccer match, Tim and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Will Tim and Michael collide?

Solution. As a first step, we can model the lines along which both Tim and Michael will travel:

Michael’s line of travel: \( f(x) = \frac{3}{2}x \), and

Tim’s line of travel: \( g(x) = \frac{1}{2}(x - 400) + 50 \).

It is an easy matter to determine where these two lines cross: Set \( f(x) = g(x) \) and solve for \( x \), getting \( x = 125 \), so the lines intersect at \( P = (125, 187.5) \).

Unfortunately, we have NOT yet determined if the runners collide. The difficulty is that we have found where the two lines of travel cross, but we have not worried about the individual locations of Michael and Tim along the lines of travel. In fact, if we compute the distance from the starting point of each person to \( P \), we find:

\[
\text{dist}(	ext{Mike}, P) = \sqrt{(0 - 125)^2 + (0 - 187.5)^2} = 225.35 \text{ feet}
\]

\[
\text{dist}(	ext{Tim}, P) = \sqrt{(400 - 125)^2 + (50 - 187.5)^2} = 307.46 \text{ feet}
\]

Since these distances are different and both runners have the same speed, Tim and Michael do not collide!

22.2 Motivation: Keeping track of a bug

Imagine a bug is located on your desktop. How can you best study its motion as time passes?

Let’s denote the location of the bug when you first observed it by \( P \). If we let \( t \) represent time elapsed since first spotting the bug (say in units of seconds), then we can let \( P(t) \) be the new location of the bug at time \( t \). When \( t = 0 \), which is the instant you first spot the bug, the location \( P(0) = P \) is the initial location. For example, the path followed by the bug might look something like the dashed path in the next Figure; we have indicated the bug’s explicit position at four future times: \( t_1 < t_2 < t_3 < t_4 \).
22.3 Examples of Parametrized Curves

How can we describe the curve in Figure 22.4? To start, let’s define a couple of new functions. Given a time \( t \), we have the point \( P(t) \) in the plane, so we can define:

\[
\begin{align*}
    x(t) &= \text{x-coordinate of } P(t) \text{ at time } t, \\
    y(t) &= \text{y-coordinate of } P(t) \text{ at time } t.
\end{align*}
\]

In other words, the point \( P(t) \) is described as

\[ P(t) = (x(t), y(t)). \]

We usually call \( x = x(t) \) and \( y = y(t) \) the coordinate functions of \( P(t) \). Also, it is common to call the pair of functions \( x = x(t) \) and \( y = y(t) \) the parametric equations for the curve. Any time we describe a curve using parametric equations, we usually call it a parametrized curve.

Given parametric equations \( x = x(t) \) and \( y = y(t) \), the domain will be the set of \( t \) values we are allowed to plug in. Notice, we are using the same set of \( t \)-values to plug into both of the equations. Describing the curve in Figure 22.4 amounts to finding the parametric equations \( x(t) \) and \( y(t) \). In other words, we typically want to come up with “formulas” for the functions \( x(t) \) and \( y(t) \). Depending on the situation, this can be easy or very hard.

22.3 Examples of Parametrized Curves

We have already worked with some interesting examples of parametric equations.

Example 22.3.1. A bug begins at the location (1,0) on the unit circle and moves counterclockwise with an angular speed of \( \omega = 2 \text{ rad/sec} \). What are the parametric equations for the motion of the bug during the first 5 seconds? Indicate, via “snapshots”, the location of the bug at 1 second time intervals.

Solution. We can use Fact 14.2.2 to find the angle swept out after \( t \) seconds: \( \theta = \omega t = 2t \) radians. The parametric equations are now easy to describe:

\[
\begin{align*}
    x &= x(t) = \cos(2t) \\
    y &= y(t) = \sin(2t).
\end{align*}
\]

If we restrict \( t \) to the domain \([0,5]\), then the location of the bug at time \( t \) is given by \( P(t) = (\cos(2t), \sin(2t)) \). We locate the bug via six one-second snapshots:
When modeling motion along a curve in the plane, we would typically be given the curve and try to find the parametric equations. We can turn this around: Given a pair of functions \( x = x(t) \) and \( y = y(t) \), let

\[
P(t) = (x(t), y(t)),
\]

which assigns to each input \( t \) a point in the \( xy \)-plane. As \( t \) ranges over a given domain of allowed \( t \) values, we will obtain a collection of points in the plane. We refer to this as the graph of the parametric equations \( \{x(t), y(t)\} \). Thus, we have now described a process which allows us to obtain a picture in the plane given a pair of equations in a common single variable \( t \). Again, we call curves that arise in this way parametrized curves. The terminology comes from the fact we are describing the curve using an auxiliary variable \( t \), which is called the describing “parameter”. In applications, \( t \) often represents time.

\[\text{Example 22.3.2. The graph of the parametric equations } x(t) = 3t \text{ and } y(t) = t + 1 \text{ on the domain } -2 \leq t \leq 2 \text{ is pictured; it is a line segment. As we let } t \text{ increase from } -2 \text{ to } 2, \text{ we can observe the motion of the corresponding points on the curve.}\]

\[\text{Figure 22.7: Observing the motion of } P.\]

22.4 Function graphs

It is important to realize that the graph of every function can be thought of as a parametrized curve. Here is the reason why: Given a function \( y = f(x) \), recall the graph consists of points \((x, f(x))\), where \( x \) runs over the allowed domain values. If we define

\[
x = x(t) = t
\]

\[
y = y(t) = f(t),
\]

then plotting the points \( P(t) = (x(t), y(t)) = (t, f(t)) \) gives us the graph of \( f \). We gain one important thing with this new viewpoint: Letting \( x = t \) increase in the domain, we now have the ability to dynamically view a point \( P(t) \) moving along the function graph. See how this works in Example 22.4.1.

\[\text{Example 22.4.1. Consider the function } y = x^2 \text{ on the domain } -2 \leq x \leq 2. \text{ As a parametrized curve, we would view the graph of } y = x^2 \text{ as all points of the form } P(t) = (t, t^2), \text{ where } -2 \leq t \leq 2. \text{ If } t \text{ increases from } -2 \text{ to } 2, \text{ the corresponding points } P(t) \text{ move along the curve as pictured:}\]

\[\text{Solution. For example, } P(-2) = (-2, (-2)^2) = (-2, 4), \text{ } P(1) = (1, 1^2) = (1, 1), \text{ etc.}\]
22.4. FUNCTION GRAPHS

Not every parametrized curve is the graph of a function. For example, consider these possible curves in the plane: The second curve from the left is the graph of a function; the other curves violate the vertical line test.

22.4.1 A useful trick

There is an approach to understanding a parametrized curve which is sometimes useful: Begin with the equation \( x = x(t) \). Solve the equation \( x = x(t) \) for \( t \) in terms of the single variable \( x \); i.e., obtain \( t = g(x) \). Then substitute \( t = g(x) \) into the other equation \( y = y(t) \), leading to an equation involving only the variables \( x \) and \( y \). If we were given the allowed \( t \) values, we can use the equation \( x = x(t) \) to determine the allowed \( x \) values, which will be the domain of \( x \) values for the function \( y = y(g(x)) \). This may be a function with which we are familiar or can plot using available software.

Example 22.4.2. Start with the parametrized curve given by the equations \( x = x(t) = 2t + 5 \) and \( y = y(t) = t^2 \), when \( 0 \leq t \leq 10 \). Find a function \( y = f(x) \) whose graph gives this parametrized curve.

Solution. Following the suggestion, we begin by solving \( x = 2t + 5 \) for \( t \), giving \( t = \frac{1}{2}(x - 5) \). Plugging this into the second equation gives \( y = \left(\frac{1}{2}(x - 5)\right)^2 = \left(\frac{1}{4}\right)(x - 5)^2 \). Conclude that \( (x, y) \) is on the parametrized curve if and only if the equation \( y = f(x) = \left(\frac{1}{2}(x - 5)\right)^2 = \left(\frac{1}{4}\right)(x - 5)^2 \) is satisfied. This is a quadratic function, so the graph will be an upward opening parabola with vertex \((5,0)\).

Since the \( t \) domain is \( 0 \leq t \leq 10 \), we get a new inequality for the \( x \) domain: \( 0 \leq \left(\frac{1}{2}\right)(x - 5) \leq 10 \). Solving this, we get \( 0 \leq x - 5 \leq 20 \), so
5 \leq x \leq 25. This means the graph of the parametrized curve is the graph of the function \( y = \left( \frac{1}{7} \right) (x - 5)^2 \), with the domain of \( x \) values \([5,25]\). Here is a plot of the graph of \( y = f(x) \); the thick portion is the parametrized curve we are studying.

### 22.5 Circular motion

We can describe the motion of an object around a circle using parametric equations. This will involve the trigonometric functions. The general setup to imagine is pictured: An object moving around a circle of radius \( r \) centered at a point \((x_c,y_c)\) in the \( xy\)-plane. The path traced out is the circle. However, the location of the object at time \( t \) will depend on a number of things:

- The starting location \( P \) of the object;
- The angular speed \( \omega \) of the object;
- The radius \( r \) and the center \((x_c,y_c)\).

We will build up to the general solution by considering two cases, the first being a special case of the second.

#### 22.5.1 Standard circular motion

As a first case to consider, assume that the center of the circle is \((0,0)\) and the starting location \( P = (r,0) \), as pictured below. If the angular speed is \( \omega \), then the angle \( \theta \) swept out in time \( t \) will be \( \theta = \omega t \); this requires that the time units in \( \omega \) agree with the time units of \( t \). We denote by \( P(t) = (x(t),y(t)) \) the \( xy\)-coordinates of the object at time \( t \). At time \( t \), we can compute the coordinates of \( P(t) = (x(t),y(t)) \) using the circular functions:

\[
\begin{align*}
x(t) &= x(t) = r \cos(\omega t) \\
y(t) &= y(t) = r \sin(\omega t).
\end{align*}
\]

This parametrizes motion starting at \( P = (r,0) \). Using the shifting technology of Chapter 9, we are led to a general description of this type of circular motion, which involves a circle of radius \( r \) centered at a point \((x_c,y_c)\); we refer to this situation as standard circular motion.

**Important Fact 22.5.1 (Standard circular motion).** Assume an object is moving around a circle of radius \( r \) centered at \((x_c,y_c)\) with a constant angular speed of \( \omega \). Assume the object begins at \( P = (x_c+r,y_c) \). Then the location of the object at time \( t \) is given by the parametric equations:

\[
\begin{align*}
x(t) &= x(t) = x_c + r \cos(\omega t) \\
y(t) &= y(t) = y_c + r \sin(\omega t).
\end{align*}
\]
Example 22.5.2. Cosmo the dog is tied to a 20 foot long tether, as in Figure 14.1. Assume Cosmo starts at the location "R" in the Figure and maintains a tight tether, moving around the circle at a constant angular speed \( \omega = \frac{\pi}{5} \) radians/second. Parametrize Cosmos motion and determine where the dog is located after 3 seconds and after 3 minutes.

Solution. Impose a coordinate system so that the pivot point of the tether is \((x_0, y_0) = (0,0)\). Since \( \omega > 0 \), Cosmo is walking counterclockwise around the circle. By (4.1.6), the location of Cosmo after \( t \) seconds is \( P(t) = (x(t), y(t)) = (20 \cos \left( \frac{\pi t}{5} \right), 20 \sin \left( \frac{\pi t}{5} \right)) \). After 3 seconds, Cosmo is located at \( P(3) = (20 \cos \left( \frac{3\pi}{5} \right), 20 \sin \left( \frac{3\pi}{5} \right)) = (-6.18, 19.02) \). After 3 minutes = 180 seconds, the location of the dog will be \( P(180) = (20 \cos \left( \frac{180\pi}{5} \right), 20 \sin \left( \frac{180\pi}{5} \right)) = (20,0) \), which is the original starting point. \( \square \)

22.5.2 General circular motion

The circular motion of an object can begin at any location \( P \) on the circle. To handle the general case, we follow an earlier idea and introduce an auxiliary relative coordinate system: The \( x_{rel} y_{rel} \)-coordinates are obtained by drawing lines parallel to the \( xy \)-axis and passing through \((x_c, y_c)\). We are using the subscript "rel" to stand for "relative". This new relative coordinate system has origin \((x_c, y_c)\) and allows us to define the initial angle \( \theta_0 \), which indexes the starting location \( P \), as pictured below:

Assume the object starts at \( P \) and is moving at a constant angular speed \( \omega \) around the pictured circle of radius \( r \). Then after time \( t \) has elapsed, the location of the object is indexed by sweeping out an angle \( \omega t \), starting from \( \theta_0 \). In other words, the location after \( t \) units of time is going to be determined by the central standard angle \( \theta_0 + \omega t \) with initial side the positive \( x_{rel} \)-axis. This means that if \( P(t) = (x(t), y(t)) \) is the location of the object at time \( t \),

\[
\begin{align*}
x &= x(t) = x_c + r \cos(\theta_0 + \omega t) \\
y &= y(t) = y_c + r \sin(\theta_0 + \omega t).
\end{align*}
\]

Notice, the case of standard circular motion is just the scenario when \( \theta_0 = 0 \) and these parametric equations collapse to those of Fact 22.5.1.

Important Fact 22.5.3 (General circular motion). Assume an object is moving around a circle of radius \( r \) centered at \((x_c, y_c)\) with a constant angular speed of \( \omega \). Assume the object begins at the location \( P \) with initial
angle $\theta_0$, as in Figure 22.14. The object location at time $t$ is given by:
\[ x = x(t) = x_c + r \cos(\theta_0 + \omega t) \quad \text{and} \quad y = y(t) = y_c + r \sin(\theta_0 + \omega t). \]

**Example 22.5.4.** A rider jumps on a merry-go-round of radius 20 feet at the pictured location. The ride rotates at the constant angular speed of $\omega = -\frac{\pi}{7}$ radians/second. The center of the platform is located 50 feet East and 50 feet North of the ticket booth for the ride. What are the parametric equations describing the location of the rider? Where is the rider after 18 seconds have elapsed? How far from the ticket booth is the rider after 18 seconds have elapsed?

**Solution.** In this example, since $\omega < 0$, the rotation is clockwise. Since the angular speed is given in radians, we need to convert the initial angle to radians as well: $218^\circ = 3.805$ radians. Impose a coordinate system so that the center of the ride is (50,50) and its radius is 20 feet. By Fact 22.5.3, the parametric equations for the rider are given by $x = x(t) = 50 + 20 \cos \left(3.805 - \frac{\pi}{7} t\right)$ and $y = y(t) = 50 + 20 \sin \left(3.805 - \frac{\pi}{7} t\right)$. The location after 18 seconds will be
\[
P(18) = (x(18), y(18)) \\
= (50 + 20 \cos(-4.273), 50 + 20 \sin(-4.273)) \\
= (41.49, 68.10).
\]

The distance from $P(18)$ to the origin is
\[
d = \sqrt{(41.49)^2 + (68.10)^2} \\
= 79.74 \text{ feet.}
\]
22.6 Exercises

Problem 22.1. Sketch the curve represented by the parametric equations. Choose four specific $t$ values and indicate the corresponding points on the curve. As $t$ moves from left to right in the domain, indicate how the corresponding points on the curve are moving.

(a) $x(t) = 1, y(t) = 4 - 2t, 0 \leq t \leq 5.$
(b) $x(t) = t - 1, y(t) = 4, 0 \leq t \leq 5.$
(c) $x(t) = t - 1, y(t) = 4 - 2t, 0 \leq t \leq 5.$
(d) $x(t) = 4 - t, y(t) = 2t - 6, 0 \leq t \leq 5.$

Problem 22.2. The population of caribou and wolves in a remote Alaskan valley is modeled by

$$x(t) = 100 \sin \left( \frac{2\pi}{7} (t - 6) \right) + 200 \quad \text{[caribou]}$$

$$y(t) = 100 \sin \left( \frac{2\pi}{7} (t - \frac{1}{2}) \right) + 200 \quad \text{[wolves]},$$

where $t$ represents years since 1970.

(a) Sketch the graphs of both population models on the same axes, for times between 1970 and 2000. Clearly label the axes on your graphs.

(b) The wolves can be regarded as predators and the caribou as prey. In a., the predator graph “lags” the prey graph. Describe what this means in words.

(c) Using sketches you made above, how often will the predator population exceed the prey population?

(d) Sketch the parametric equations $A(t) = \{x(t), y(t)\}$ in the xy-coordinate system for the time domain $0 \leq t \leq 7.$

(e) Using your parametric sketch from the previous question, plot the point $A(0) = \{x(0), y(0)\}$. Describe the motion of $A(t)$ around the plot for $0 \leq t \leq 7$ and interpret what is happening to the predator/prey population.

(f) The situation described in a.-e. is “cyclic”; i.e. the population of each species cycles sinusoidally. Suppose that the population models were different functions $x^*(t)$ and $y^*(t)$ and that the graph of $A^*(t) = \{x^*(t), y^*(t)\}$ is as pictured below. Explain what is happening. (Note: $A(0)$ is the right-most endpoint of the curve.)

Problem 22.3. Return to Example 22.3.1 on Page 305. What are the coordinates of the bug in the six “snapshots” in the solution? When will the bug first cross from the first to second quadrant? When will the bug first have x-coordinate $= -0.2$?

Problem 22.4. Lee has been boasting all year about his basketball skill and Allyson can’t stand it anymore. So, she has challenged Lee to a game of one-on-one. With 2 seconds to go, Allyson is leading 14-12 and Lee res a desperation three. Impose coordinates with Lee’s feet at the origin and use units of feet on each axis. The path of the ball is described by the parametric equations: $x(t) = 28.925t, y(t) = -16t^2 + 34.472t + 6.$

(a) How high is the ball when Lee fires his shot?

(b) When is the ball directly over Allyson?
(c) Allyson can jump and block a ball 9.5 feet above the floor. Can Allyson block Lee’s shot?

(d) Where and when is the ball 22 feet above the floor?

(e) Where and when does the ball reach its highest point above the floor?

(f) Who wins the game?

(g) Solve the equation $x = x(t)$ for $t$ in terms of $x$, then plug this into the equation for $y = y(t)$. Sketch the graph of the resulting function $y = f(x)$; this is the path of Lee’s shot.

Problem 22.5. A six foot long rod is attached at one end $A$ to a point on a wheel of radius 2 feet, centered at the origin. The other end $B$ is free to move back and forth along the $x$-axis. The point $A$ is at $(2,0)$ at time $t = 0$, and the wheel rotates counterclockwise at 3 rev/sec.

(a) As the point $A$ makes one complete revolution, indicate in the picture the direction and range of motion of the point $B$.

(b) Find the coordinates of the point $A$ as a function of time $t$.

(c) Find the coordinates of the point $B$ as a function of time $t$.

(d) What is the $x$-coordinate of the point $B$ when $t = 1$? You should be able to find this two ways: with your function from part (c), and using some common sense (where is point A after one second?).

(e) Find the first two times when the $x$ coordinate of the point $B$ is 5.

Problem 22.6. A ferris wheel of radius 100 feet is rotating at a constant angular speed $\omega$ counterclockwise. Using a stopwatch, the rider finds it takes 3.4 seconds to go from the lowest point on the ride to a point $Q$, which is level with the top of a 44 ft pole. Assume the lowest point of the ride is 3 feet above ground level.

(a) What is the angular speed $\omega$? How fast is the rider moving in mph?

(b) Find parametric equations for the motion of a rider on the wheel, assuming the rider begins at the lowest point on the wheel.

(c) Due to a malfunction, the ride abruptly stops 35 seconds after it began. To the nearest foot, how high above the ground is the rider?

Problem 22.7. Cherie is running clockwise around a circular track of radius 300 feet. She starts at the location pictured running with an angular speed of $\frac{5}{9}$ RPM clockwise. Julie stands at the pictured location, 100 feet from the center of the circle.

(a) When will Cherie first reach the location on the track closest to Julie?

(b) Where is Cherie located in 40 seconds? (i.e. find her coordinates). Indicate this point in your picture and label it as $P(40)$. 
22.6. EXERCISES

(c) How far has Cherie traveled (distance she has run) in 40 seconds?

(d) Let \( d(t) \) be the function that calculates the distance between Cherie and Julie at time \( t \) seconds.

(d1) If the domain is taken to be the time required for Cherie to complete one revolution, what is the domain and range of \( d(t) \)?

(d2) Write down a formula to calculate \( d(t) \).

(e) Assume that Julie is instead located at the position \((0,a)\), where \( 0 \leq a \leq 300 \) is a given constant. (So part (d) was the case when \( a = 100 \).) Find a function \( d_a(t) \) that computes the distance between Julie and Cherie at time \( t \). Your formula will involve \( t \) and \( a \) and should collapse to the formula for \( d(t) \) in (d) if you set \( a = 100 \).

Problem 22.8. In the pictures below, a bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, find parametric equations describing the location of the bug at time \( t \) seconds.