Radian Measure

The **radian measure** of an angle at the center of the unit circle equals the length of the arc that the angle cuts from the unit circle.

**Arc Length**
The length of an arc $s$ subtended on a circle of radius $r$ by a central angle of measure $\theta$ is: $s = r\theta$.

**You try:**
The angle (in radians) lies at the center of a circle and subtends an arc of the circle. Find the missing angle measure, circle radius, or arc length.

Graphs of Trigonometric Functions

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable (radians) by $x$ instead of $\theta$. Calculus is built on the assumption that all angles are measured in radians, unless degrees or some other unit is stated explicitly. When the angle $\pi/3$ is talked about, it is meant $\pi/3$ radians (which is $60^\circ$) not $\pi/3$ degrees. *When you do calculus, keep your calculator in radian mode.*

<table>
<thead>
<tr>
<th>Angle</th>
<th>Radius</th>
<th>Arc Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $5\pi/6$</td>
<td>2</td>
<td>$=$</td>
</tr>
<tr>
<td>2. $175^\circ$</td>
<td>$=$</td>
<td>10</td>
</tr>
<tr>
<td>3. $=$</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>4. $=$</td>
<td>6</td>
<td>$3\pi/2$</td>
</tr>
</tbody>
</table>

**Domain:** $\infty < x < \infty$
**Range:** $-1 \leq y \leq -1$
**Period:** $2\pi$

**Domain:** $-\infty < x < \infty$
**Range:** $-1 \leq y \leq -1$
**Period:** $2\pi$

**Domain:** $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$
**Range:** $y \leq -1$ and $y \geq 1$
**Period:** $2\pi$
Periodicity
When an angle of measure \( \theta \) and an angle of measure \( \theta + 2\pi \) are in standard position their terminal rays coincide. The two angles therefore have the same trigonometric function values:

\[
\begin{align*}
\cos(\theta + 2\pi) &= \cos \theta \\
\sin(\theta + 2\pi) &= \sin \theta \\
\tan(\theta + 2\pi) &= \tan \theta \\
\sec(\theta + 2\pi) &= \sec \theta \\
\csc(\theta + 2\pi) &= \csc \theta \\
\cot(\theta + 2\pi) &= \cot \theta
\end{align*}
\]

Similarly, \( \cos(\theta - 2\pi) = \cos \theta \), \( \sin(\theta - 2\pi) = \sin \theta \), and so on. We see the values of the trigonometric functions repeat at regular intervals. We describe this behavior by saying that the six basic trigonometric functions are periodic.

**DEFINITION** Periodic Functions, Period
A function \( f(x) \) is periodic if there is a positive number \( p \) such that \( f(x + p) = f(x) \) for every value of \( x \). The smallest such value of \( p \) is the period of \( f \).

Even and Odd Trigonometric Functions
The graphs of \( \cos(x) \) and \( \sec(x) \) are even functions because their graphs are symmetric about the \( y \)-axis. The other four basic trigonometric functions are odd. Recall that for even functions \( f(-x) = f(x) \) and for odd functions \( f(-x) = -f(x) \).

Recall from precalculus the definitions of your six trigonometric functions. When an angle of measure \( \theta \) is placed in standard position at the center of a circle of radius \( r \)

\[
\begin{align*}
sine: \quad \sin \theta &= \frac{y}{r} \\
cosecant: \quad \csc \theta &= \frac{r}{y} \\
cosine: \quad \cos \theta &= \frac{x}{r} \\
secant: \quad \sec \theta &= \frac{r}{x} \\
tangent: \quad \tan \theta &= \frac{y}{x} \\
cotangent: \quad \cot \theta &= \frac{x}{y}
\end{align*}
\]

**You try**
Find all the trigonometric values of \( \theta \) with the given conditions:

i) \( \sin \theta = -3/5 \) and \( \tan \theta < 0 \) 
ii) \( \cos \theta = -15/17 \) and \( \sin \theta > 0 \)

iii) \( \tan \theta = -1 \) and \( \sin \theta < 0 \) 
iv) \( \cot \theta = 1 \) and \( \cos \theta < 0 \)
Transformations of Trigonometric Graphs
The rules for shifting, stretching, shrinking, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.

\[
y = af \left( b \left( x - c \right) \right) + d
\]

Horizontal stretch or shrink:
reflection about y-axis

Vertical stretch or shrink:
reflection about x-axis

Vertical shift

Horizontal shift

The general sine function or **sinusoid** can be written in the form

\[
f(x) = A \sin \left[ \frac{2\pi}{B} \left( x - C \right) \right] + D
\]

where \( |A| \) is the **amplitude**, \( |B| \) is the **period**, \( |C| \) is the **horizontal shift**, and \( D \) is the **vertical shift**.

**Example: Graphing a Trigonometric Function**
Determine the (a) period, (b) domain, (c) range, and (d) draw the graph of the function \( y = 3 \cos (2x - \pi) + 1 \).

**Solution**
We can write the function in the form \( y = 3 \cos \left( 2 \left( x - \frac{\pi}{2} \right) \right) + 1 \).

(a) The period is given by \( \frac{2\pi}{B} \), where \( \frac{2\pi}{B} = 2 \). The period is \( \pi \).
(b) The domain is \((-\infty, \infty)\).
(c) The graph is a basic cosine curve with amplitude 3 that has been shifted up 1 unit. Thus the range is \([-2, 4]\).
(d) The graph has been shifted to the right \( \frac{\pi}{2} \) units. The graph is shown below together with the graph of \( \cos(x) \). Notice that four periods of \( y = 3 \cos (2x - \pi) + 1 \) are drawn in the window \([-2\pi, 2\pi]\) by \([-4, 6]\).

**You try**
Determine (a) period, (b) the domain, (c) the range, and (d) draw the graph of the function.

i) \( y = -3 \tan (3x + \pi) + 2 \)

ii) \( y = 2 \sin \left( 2x + \frac{\pi}{3} \right) \)
Specify (a) the period, (b) the amplitude, and (c) the viewing window that is shown.

\[
\begin{align*}
\text{i)} & \quad y = 1.5 \sin 2x \\
\text{ii)} & \quad y = 2 \cos 3x
\end{align*}
\]

Inverse Trigonometric Functions
None of the six basic trigonometric functions graphed on the first page are one-to-one, therefore these functions do not have inverses that are functions. However, in each case the domain can be restricted to produce a new function that does have an inverse, as illustrated below.

Example  Restricting the Domain of the Sine
Show that the function \( y = \sin x, \ -\pi / 2 \leq x \leq \pi / 2 \), is one-to-one, and graph its inverse.

Solution
The figure below shows this restricted sine function using the parametric equations

\[
x_1 = t, \quad y_1 = \sin t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.
\]

This restricted sine function is one-to-one because it does not repeat any output values. it therefore has an inverse, which we graph to the right by interchanging the ordered pairs using the parametric equations

\[
x_2 = \sin t, \quad y_2 = t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.
\]

You try
Show that the function is one-to-one and graph its inverse.

\[
\begin{align*}
\text{i)} & \quad y = \cos x, \quad 0 \leq x \leq \pi \\
\text{ii)} & \quad y = \tan x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
\end{align*}
\]
The inverse of the restricted sine function above is called the *inverse sine function*. The inverse sine of \( x \) is the angle whose sine is \( x \). It is denoted by \( \sin^{-1} x \) or \( \arcsin x \). Either notation is read “arcsine of \( x \)” or he inverse of sine of \( x \)”.

The domains of the other basic trigonometric functions can also be restricted to produce a function with an inverse. The domains and ranges of the resulting inverse functions become parts of their definitions.

### DEFINITIONS Inverse Trigonometric Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos^{-1} x )</td>
<td>(-1 \leq x \leq 1)</td>
<td>(0 \leq y \leq \pi)</td>
</tr>
<tr>
<td>( y = \sin^{-1} x )</td>
<td>(-1 \leq x \leq 1)</td>
<td>(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})</td>
</tr>
<tr>
<td>( y = \tan^{-1} x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
<td>(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})</td>
</tr>
<tr>
<td>( y = \sec^{-1} x )</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( y = \csc^{-1} x )</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( y = \cot^{-1} x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
<td>(0 &lt; y &lt; \pi)</td>
</tr>
</tbody>
</table>

The graphs of the inverse trig functions are below. Match the graph with its name.

a) \( y = \cos^{-1} x \)  
   b) \( y = \sin^{-1} x \)  
   c) \( y = \tan^{-1} x \)  
   d) \( y = \sec^{-1} x \)  
   e) \( y = \csc^{-1} x \)  
   f) \( y = \cot^{-1} x \)

Example: Using the Inverse Trigonometric Functions

Solve for \( x \).

(a) \( \sin x = 0.7 \) in \( 0 \leq x < 2\pi \)  
(b) \( \tan x = -2 \) in \( -\infty < x < \infty \)

**Solution**

(a) Notice that \( x = \sin^{-1} (0.7) \approx 0.775 \) is in the first quadrant, so 0.775 is one solution of this equation. The angle \( \pi - x \) is in the second quadrant and has sine equal to 0.7. Thus two solutions in this interval are

\[ \sin^{-1} (0.7) \approx 0.775 \quad \text{and} \quad \pi - \sin^{-1} (0.7) \approx 2.366. \]
(b) The angle \( \tan^{-1}(-2) \approx -1.107\) is in the fourth quadrant, so 0.775 is the only solution to this equation in the interval \(-\pi/2 < x < \pi/2\) where \( \tan x \) is one-to-one. Since \( \tan x \) is periodic with period \( \pi \), the solutions to this equation are of the form
\[
\tan^{-1}(-2) = k\pi - 1.107 \quad \text{where} \quad k \text{ is any integer.}
\]

**Practice (use your own paper)**

1. Specify (a) the period, (b) the amplitude, and (c) the viewing window that is shown.

   i) \( y = -4\sin\frac{\pi}{3}x \)

   ![Graph](image1)

   ii) \( y = 5\sin\frac{x}{2} \)

   ![Graph](image2)

2. Solve the equation in the specified interval.

   i) \( \tan x = 2.5, \quad 0 \leq x \leq 2\pi \)

   ii) \( \csc x = 2, \quad 0 < x < 2\pi \)

   iii) \( \sec x = -3, \quad -\pi \leq x < \pi \)

3. Use the given information to find the values of the six trigonometric functions at the angle \( \theta \).

   Give exact answers.

   i) \( \theta = \sin^{-1}\left(\frac{8}{17}\right) \)

   ii) \( \theta = \tan^{-1}\left(-\frac{5}{12}\right) \)

   iii) The point \( P(-3,4) \) is on the terminal side of \( \theta \).

   iv) The point \( P(-2,2) \) is on the terminal side of \( \theta \).

4. Evaluate the expression without your calculator.

   i) \( \sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right) \)

   ii) \( \tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right) \)

5. Find a possible formula for each graph.

   i) \([\pi, \pi] \) by \([-5,5]\)

   ![Graph](image3)

   ii) \([0, 2\pi] \) by \([-9,9]\)

   ![Graph](image4)

   iii) \([0.8] \) by \([0.6]\)

   ![Graph](image5)
6. Find an exact solution and a decimal approximation to the equation.
   i) \(2 - 5\sin(3x)\)  
   ii) \(1 = 8\cos(2x + 1) - 3\)

7. A compact disk spins at a rate of 200 to 500 revolutions per minute. What are the equivalent rates measured in radians per second?

8. When a car’s engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?

9. What is the difference between \(\sin x^2\), \(\sin^2 x\), and \(\sin(\sin x)\)? Express each of the three as a composition.

10. On the graph of \(y = \sin x\), points \(P\) and \(Q\) are at consecutive lowest and highest points. Find the slope of the line through \(P\) and \(Q\).

11. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, \(y\) meters, is given as a function of time, \(t\), in hours since midnight by
   \[y = D = A\cos\left(B(t - C)\right)\].
   (a) What is the physical meaning of \(D\)?
   (b) What is the value of \(A\)?
   (c) What is the value of \(B\)? Assume the time between successive high tides is 12.4 hours.
   (d) What is the physical meaning of \(C\)?

12. A population of animals oscillates sinusoidally between a low of 700 on January 1 and a high of 900 on July 1.
   (a) Graph the population against time.
   (b) Find a formula for the population as a function of time, \(t\), in months since the start of the year.

13. Find the area of the trapezoid cross-section of the irrigation canal shown below.

14. A baseball hit at an angle of \(\theta\) to the horizontal with initial velocity \(v_0\) has horizontal range, \(R\), given by \(R = \frac{v_0^2}{g} \sin(2\theta)\). Here \(g\) is the acceleration due to gravity. Sketch \(R\) as a function of \(\theta\) for \(0 \leq \theta \leq \pi/2\).
   What angle gives the maximum range? What is the maximum range?

15. The desert temperature, \(H\), oscillates daily between 40°F at 5 am and 80°F at 5 pm. Write a possible formula for \(H\) in terms of \(t\), measured in hours from 5 am.