One-to-One Functions
As you know, a function is a rule that assigns a single value in its range to each point in its domain. Some functions assign the same output to more than one input. For example, \( f(x) = x^2 \) assigns the output 4 to both 2 and -2. Other functions never output a given value more than once. For example, the curves of different numbers are always different. If each output value of a function is associated with exactly one input value, the function is one-to-one.

**DEFINITION One-to-One Function**
A function \( f(x) \) is one-to-one on a domain \( D \) if \( f(a) \neq f(b) \) whenever \( a \neq b \).

The graph of a one-to-one function \( y = f(x) \) can intersect any horizontal line at most once (the horizontal line test). If it intersects such a line more than once it assumes the same \( y \)-value more than once, and is therefore not one-to-one.

**EXAMPLE Using the Horizontal Line Test**
Determine whether the functions are one-to-one.

(a) \( f(x) = |x| \)
(b) \( g(x) = \sqrt{x} \)

**Solution**
(a) As shown below, each horizontal line \( y = c, c > 0 \), intersects the graph of \( f(x) = |x| \) twice. So \( f \) is not one-to-one.

(b) As shown below, each horizontal line intersects the graph of \( g(x) = \sqrt{x} \) either once or not at all. The function \( g \) is one-to-one.

Functions that are not one-to-one are called many-to-one because there can be more than one input for each distinct output, or the same output can be assigned to different inputs.
Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined by reversing a one-to-one function \( f \) is the inverse of \( f \). The function in the tables below are inverses of one another. The symbol for the inverse of \( f \) is \( f^{-1} \), read "\( f \) inverse. The -1 in \( f^{-1} \) is not an exponent; \( f^{-1} \) does not mean \( 1/f(x) \).

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<th>Time ( x ) (hours)</th>
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<th>Time ( x ) (hours)</th>
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As is suggested by the tables above, composing a function with its inverse in either order sends each output back to the input from which it came. In other words, the result of composing a function and its inverse in either order is the identity function, the function that assigns each number to itself. This gives a way to test whether two functions \( f \) and \( g \) are inverses of one another. Compute \( f \circ g \) and \( g \circ f \). If \( (f \circ g)(x) = (g \circ f)(x) = x \), then \( f \) and \( g \) are inverses of one another; otherwise they are not.

EXPLORATION Testing for Inverses Graphically

For each of the function pairs below,
(a) Graph \( f \) and \( g \) together in a square window.
(b) Graph \( f \circ g \).
(c) Graph \( g \circ f \). What can you conclude from the graphs?

1. \( f(x) = x^3 \), \( g(x) = x^{1/3} \)
2. \( f(x) = x \), \( g(x) = 1/x \)
3. \( f(x) = 3x \), \( g(x) = x/3 \)
4. \( f(x) = e^x \), \( g(x) = \ln x \)

Finding Inverses

Read the UW precalculus section on inverses, the concept of inverses, and the graphical ideas of inverses. Be sure you understand inverses completely. Try some of the UW questions, and check your answers.

Writing \( f^{-1} \) as a Function of \( x \)

1. Solve the equation \( y = f(x) \) for \( x \) in terms of \( y \).
2. Interchange \( x \) and \( y \). The resulting formula will be \( y = f^{-1}(x) \).

EXAMPLE Find the Inverse Function

Show that the function \( y = f(x) = -2x + 4 \) is one-to-one and find its inverse function.

Solution

Every horizontal line intersects the graph of \( f \) exactly once, so \( f \) is one-to-one and has an inverse.

Step 1: Solve for \( x \) in terms of \( y \):
\[ y = -2x + 4 \]
\[ x = \frac{1}{2} y + 2 \]

Step 2: Interchange \( x \) and \( y \):
\[ y = \frac{1}{2} x + 2 \]
The inverse of the function \( f(x) = -2x + 4 \) is the function \( f^{-1}(x) = -(1/2)x + 2 \). We can verify that both composites are the identity function.

\[
f^{-1}(f(x)) = \frac{1}{2}(-2x + 4) + 2 = x - 2 + 2 = x
\]

\[
f(f^{-1}(x)) = -2\left(\frac{1}{2}x + 2\right) + 4 = x - 4 + 4 = x
\]

**You try**

Find \( f^{-1} \) and verify that \( (f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x \).

(a) \( f(x) = 2x + 3 \)

(b) \( f(x) = x^2 + 1, \ x \leq 0 \)

We can use parametric graphing to graph the inverse of a function without find an explicit rule for the inverse, as illustrated below.

**EXAMPLE  Graphing the Inverse Parametrically**

(a) Graph the one-to-one function \( f(x) = x^2, \ x \geq 0 \), together with its inverse and the line \( y = x, \ x \geq 0 \).

(b) Express the inverse of \( f \) as a function of \( x \).

**Solution**

(a) We can graph the three functions parametrically as follows:

Graph of \( f \): \( x_1 = t, \ y_1 = t^2, \ t \geq 0 \)

Graph of \( f^{-1} \): \( x_2 = t^2, \ y_2 = t \)

Graph of \( y = x \): \( x_3 = t, \ y_3 = t \)

(b) Next we find a formula for \( f^{-1}(x) \).

**Step 1**

Solve for \( x \) in terms of \( y \):

\[
y = x^2
\]

\[
\sqrt{y} = \sqrt{x^2}
\]

\[
\sqrt{y} = |x| = x, \text{ because } x \geq 0
\]

**Step 2**

Interchange \( x \) and \( y \),

\[
\sqrt{x} = y
\]

Thus \( f^{-1}(x) = \sqrt{x} \).
Practice Problems

1. Find \( f^{-1} \) and verify that \((f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x\). Then find the domain and ranges of both \( f(x) \) and \( f^{-1}(x) \). What do you notice?
   
   (a) \( f(x) = x^2 + 2x + 1, \ x \geq -1 \)  
   (b) \( f(x) = \frac{2x + 1}{x + 3} \)

2. Use parametric graphing to graph \( f, f^{-1}, \) and \( y = x \).

   9a) \( f(x) = \sin^{-1} x \)  
   (b) \( f(x) = 2^{-x} \)

3. Solve \( e^x + e^{-x} = 3 \) algebraically (hint: think quadratic).

4. For \( g(x) = x^2 + 2x + 3 \) find and simplify:
   
   (a) \( g(2 + \delta) \)  
   (b) \( g(2) \)  
   (c) \( g(2 + \delta) - g(2) \)

5. For \( f(n) = 3n^2 - 2 \) and \( g(n) = n + 1 \), find and simplify:
   
   (a) \( f(n) + g(n) \)  
   (b) \( f(n)g(n) \)  
   (c) the domain of \( f(n)/g(n) \)  
   (d) \( f(g(n)) \) and \( g(f(n)) \)

6. Let \( p \) be the price of an item and \( q \) be the number of items sold at that price, where \( q = f(p) \). What do the following quantities mean in terms of prices and quantities sold?

   (a) \( f(25) \)  
   (b) \( f^{-1}(30) \)

7. Are the functions even, odd, or neither?

   (a) \( f(x) = x^6 + x^3 + 1 \)  
   (b) \( f(x) = x^3 + x^2 + x \)

8. Decide if the function \( f \) is invertible.
   (a) \( f(t) \) is the number of customers in Macy’s department store at \( t \) minutes past noon on December 18, 2000.
   (b) \( f(x) \) is the volume in liters of \( x \) kg of water at \( 4^\circ C \).

9. Use the graphs above to answer the following.

   (a) Estimate \( f(g(1)) \).  
   (b) Estimate \( g(f(2)) \)

   (c) Graph \( f(f(x)) \)  
   (d) Graph \( g(f(x)) \)
10. The cost of producing $q$ articles is given by the function $C = f(q) = 100 + 2q$.
   (a) Find a formula for the inverse function.
   (b) Explain in practical terms what the inverse function tells you.

11. Complete the following table with values for the function $f$, $g$, and $h$, given that:

   (a) $f$ is symmetric about the $y$-axis.
   (b) $g$ is symmetric about the origin.
   (c) $h$ is the composition $h(x) = g(f(x))$.

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12. A tree of height $y$ meters has, on average, $B$ branches, where $B = y - 1$. Each branch has, on average, $n$ leaves where $n = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.