**Week #24: March 2 – March 6**

**Objectives:** The students will learn that an exponential function is a function with the general form $y = ab^x$, $a \neq 0$, with $b > 0$, and $b \neq 1$ and that $b$ is a constant, the exponent $x$ is the independent variable with a domain the set of real numbers. The student will learn to graph exponential functions, determine the domain, range, and $y$-intercept, understand what the asymptote is and how it affects the function, be able to contrast exponential growth vs. exponential decay, use the equation and $A(t) = a(1+r)^t$ apply this knowledge to real world problems (in particular, compound interest problems). The students will also learn the factor $a$ in $y = ab^x$ can stretch or compress and possibly reflect the graph of the parent function $y = b^x$ and the function $y = ab^x$, $a > 0$, $b > 1$ models exponential growth, while $y = ab^x$ models exponential decay if $0 < b < 1$, as well as how transformation affects domain, range, and asymptotes of the function, learn about the number $e$, be able to graph $y = e^x$ and use $e$ in continuous compounding interest problems.

<table>
<thead>
<tr>
<th>Day</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>Read Lesson 7-1; Exploring Exponential Models</td>
</tr>
<tr>
<td></td>
<td>Complete Practice Worksheet 7-1</td>
</tr>
<tr>
<td>Block Day</td>
<td>Read Lesson 7-2; Properties of Exponential Functions</td>
</tr>
<tr>
<td></td>
<td>Complete Practice Worksheet 7-2</td>
</tr>
<tr>
<td></td>
<td>Complete Enrichment Worksheet 7-2</td>
</tr>
<tr>
<td>Thursday</td>
<td>Read Lesson 7-3; Logarithmic Functions as Inverses</td>
</tr>
<tr>
<td></td>
<td>Begin Practice Worksheet 7-3</td>
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<tr>
<td>Friday</td>
<td>Complete Practice Worksheet 7-3</td>
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<tr>
<td></td>
<td>Complete Enrichment Worksheet 7-3</td>
</tr>
<tr>
<td></td>
<td>Complete Fitting Curves to Data Concept Byte #1, 2, and 3</td>
</tr>
</tbody>
</table>

I will collect all of these assignments and the notes at the beginning of class, Monday, March 9th.

**Your weekly homework package will be graded as follows:**

A homework package that earns a perfect score of 10/10 points meets the following criteria:

- All problems have been attempted.
- Work is shown step-by-step for every problem.
- A thorough explanation is provided for all problems requiring an explanation.
- All work and explanations are legible.
- All applicable graphs, drawings, and proofs are completed.
- The assignment has been visibly corrected in pen.
- All notes and problems from the class are completed.
- The weekly package was turned in on time.
KEY TERMS
Exponential Function  Exponential Growth  Exponential Decay
Asymptote  Growth Factor  Decay Factor

7-1 Solve It!

This is a famous puzzle. Variations of it show up in many video games.

Getting Ready!

The rules:
- There are two stacks of rings. One has 5 rings. The puzzle is to move the stack of 5 rings to another.
- The rules of the game are that you may not place a ring on top of a smaller ring.
- You can move only one ring at a time.
- The fewest number of moves needed?
- How many moves are needed for 10 rings? 20 rings? Explain.

How many times does the original bottom ring move?

31; 1023; 1048575; the number of moves follows the pattern $2^n-1$.

How many times does the smallest ring move from its original position to its final position if there are $n$ rings? Explain. $2^{n-1}$ times; the smallest ring gets every second move. In between moving it, there is only one legal move that is not moving it again.

Is there a post you have to move the smallest ring to as the first move if you want the final stack on a specific post? Explain. Yes; for $n$ odd, move it to the post where you want the stack, and for $n$ even, move it to the post where you do not want the stack.

The number of moves needed for additional rings in the Solve It! suggests a pattern that approximates repeated multiplication.

ESSENTIAL UNDERSTANDING – You can represent repeated multiplication with a function of the form $y=ab^x$ where $b$ is a positive number other than 1.

An exponential function is a function with the general form $y=ab^x, a \neq 0$, with $b > 0$, and $b \neq 1$. In an exponential function, the base $b$ is constant. The exponent $x$ is the independent variable in the domain the set of real numbers.
EXAMPLE 1 – Graph of an Exponential Function
What is the graph of $y = 3.1^x$?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3.1</td>
</tr>
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<td>-3</td>
<td>0.031</td>
</tr>
<tr>
<td>2</td>
<td>9.61</td>
</tr>
</tbody>
</table>

Use calculator after building basic table.

TRY IT –
What is the graph of $y = 4^x$?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.25</td>
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<tr>
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<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
</tr>
</tbody>
</table>

Use calculator.

Two types of exponential behavior are exponential growth and exponential decay.

For exponential growth, as the value of $x$ increases, the value of $y$ increases. For exponential decay, as the $x$ values increases, the value of $y$ decreases approaching zero.

The exponential functions shown here are asymptotic to the $x$-axis. An asymptote is a line that a graph approaches as $x$ or $y$ increases in absolute value.
**Concept Summary** Exponential Functions

For the function $y = ab^x$,

- if $a > 0$ and $b > 1$, the function represents exponential growth.
- if $a > 0$ and $0 < b < 1$, the function represents exponential decay.

In either case, the $y$-intercept is $(0, a)$, the domain is all real numbers, the asymptote is $y = 0$, and the range is $y > 0$.

**EXAMPLE 2** – Identifying Exponential Growth and Decay

Identify $y = 0.7^x$ as an example of exponential growth or decay. What is the $y$-intercept?

```
decay ; (0,1)
```

**TRY IT** –

Identify $y = 3(4)^x$ as an example of exponential growth or decay. What is the $y$-intercept?

```
growth ; (0,3)
```

For exponential growth, $y = ab^x$, with $b > 1$, the value $b$ is the **growth factor**. A quantity that exhibits exponential growth increases by a constant percentage each time period. The percentage increase $r$, written as a decimal, is the *rate of increase* or *growth rate*.

For exponential growth $b = 1 + r$.

For exponential decay, $0 < b < 1$ and $b$ is the **decay factor**. The quantity decreases by a constant percentage each time period. The percentage decrease, $r$, is the *rate of decay*. Usually a rate of decay is expressed as a negative quantity, so $b = 1 + r$.

**Key Concept** Exponential Growth and Decay

You can model exponential growth or decay with this function.

```
Amount after $t$ time periods = $a(1 + r)^t$
Rate of growth ($r > 0$) OR decay ($r < 0$)
Initial amount = $a$
Number of time periods = $t$
```

For growth or decay to be exponential, a quantity changes by a fixed percentage each time period.
EXAMPLE 3 – Modeling Exponential Growth
You buy a savings bond for $25 that pays a yearly interest rate of 4.2%. What will the savings bond be worth after fifteen years?

\[
A(t) = a(1+r)^t \\
= 25(1+.042)^{15} \\
= \$ 46.34
\]

TRY IT –
Suppose you invest $500 in a savings account that pays 3.5% annual interest. How much will be in the account after five years?

\[
A(t) = P(1+r)^t \\
= 500(1+.035)^5 \\
= \$ 593.84
\]

EXAMPLE 4 – Using Exponential Growth
You open a savings account that pays 4.5% annual interest. If your initial investment is $300 and you make no additional deposits or withdrawals, how many years will it take for the account to grow to at least $500?

\[
\frac{500}{300} = (1+.045)^t \\
\frac{5}{3} = (1+.045)^t \\
\text{for now, use just the}
\]

TRY IT –
Suppose you invest $500 in a savings account that pays 3.5% annual interest. When will the account contain $650?

\[
A(t) = 500(1+.035)^t \\
\text{by the 8th yr}
\]

Explain how you can tell whether \( y = ab^x \) represents exponential growth or exponential decay.

\[
\text{if } a > 0 \text{ and } b > 1, \text{ exp. growth} \\
\text{if } a > 0 \text{ and } 0 < b < 1, \text{ exp. decay}
\]
Exponential functions are often discrete. In example 4, interest is only paid once a year. So the graph consists of individual points corresponding to \( t = 1, 2, 3 \) and so on. It is not continuous. Both the table and the graph show that there is never exactly $1500 in the account and that the account will not contain more than $1500 until the ninth year.

![Graph showing account balance vs. year](image)

To model a discrete situation using an exponential function of the form \( y = ab^t \), you need to find the growth or decay factor \( b \). If you know \( y \)-values for two consecutive \( x \)-values, you can find the rate of change \( r \), and then find \( b \) using \( r = \frac{(y_2 - y_1)}{y_1} \)

\[ b = 1 + r. \]

**EXAMPLE 5 – Writing an Exponential Function**

The initial value of a car is $30,000. After one year, the value of the car is $20,000. Estimate the value of the car after 5 years.

\[ y = 30,000 \left( \frac{2}{3} \right)^5 \]

\[ = 18,500 \]

\[ r = \frac{20,000 - 30,000}{30,000} \]

\[ r = -\frac{1}{3} \]

\[ b = 1 + (-\frac{1}{3}) = \frac{2}{3} \]

**TRY IT**

The table shows the world population of the Iberian lynx in 2003 and 2004. If this trend continues and the population is decreasing exponentially, how many Iberian lynx will there be in 2014?

\[ \frac{120 - 150}{150} = -0.2 \]

\[ b = 1 + (-0.2) \]

\[ = 0.8 \]

\[ y = ab^x \rightarrow y = 150 \left( 0.8 \right)^t \]

\[ \approx 13 \text{ Iberian lynx} \]
7-1 Lesson Quiz

1. What is the graph of \( y = \left( \frac{1}{3} \right)^x \)?

2. Identify \( y = 3(1.2)^x \) as an example of exponential growth or decay. What is the \( y \)-intercept?

3. You deposit $3000 in an account that pays 5% annual interest. What is the balance after 2 years?

4. You invest $75 in a savings account that pays 2% annual interest. If you make no additional deposits or withdrawals, how many years will it take for the account to grow to at least $100?

5. Do you UNDERSTAND? There were 50,000 bacteria in a petri dish yesterday at noon, and 40,000 bacteria at noon today. If the trend continues, on what day at noon can you expect to find less than 5,000 bacteria?
Graph each function.

1. \( y = (0.3)^x \)
   
<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1.11</td>
</tr>
<tr>
<td>-1</td>
<td>0.33</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

2. \( y = 2 \left( \frac{1}{3} \right)^x \)

3. \( s(t) = 2.5^t \)

Without graphing, determine whether the function represents exponential growth or exponential decay. Then find the \( y \)-intercept.

4. \( y = 0.99 \left( \frac{1}{3} \right)^x \) exponential decay
   \( \text{y-int: } (0, 0.99) \quad 0 < b = \frac{1}{3} < 1 \)
   \( y = 0.99 \left( \frac{1}{3} \right)^0 \) \( y = 0.99 \)

5. \( y = 185 \left( \frac{3}{5} \right)^x \) exponential growth
   \( \text{y-int: } (0, 185) \quad b = \frac{3}{5} > 1 \)
   \( y = 185 \left( \frac{3}{5} \right)^0 \) \( y = 185 \)

6. \( f(x) = 0.25 \left( 1.05 \right)^x \) exponential growth
   \( \text{y-int: } (0, 0.25) \quad b = 1.05 > 1 \)
   \( f(x) = 0.25 \left( 1.05 \right)^0 \) \( f(x) = 0.25 \)

7. Suppose you deposit $1500 in a savings account that pays interest at an annual rate of 6%. No money is added or withdrawn from the account.
   a. How much will be in the account after 5 years?
      \( A = 1500 \left( 1 + 0.06 \right)^5 \)
      \( A = 2007.39 \)
   b. How much will be in the account after 20 years?
      \( A = 1500 \left( 1 + 0.06 \right)^{20} \)
      \( A = 4810.70 \)
   c. How many years will it take for the account to contain $2500?
      using graphing calculator, point of intersection is \approx 9 \text{ yrs}
   d. How many years will it take for the account to contain $4000?
      using graphing calculator, point of intersection is \approx 17 \text{ yrs}
8. Write an exponential function to model the situation for a population of 752,000 that decreases 1.4% per year for 18 years. What is the population after 18 years?

\[ f(x) = 752,000 \left(1 - 0.014\right)^x \]
\[ f(18) = 752,000 \left(0.986\right)^{18} \approx 583,446 \text{ people} \]

For each annual rate of change, find the corresponding growth or decay factor.

9. +45% = 1.45

10. -40% = 0.60

11. +28% = 1.28

12. -5% = 0.95

13. In 2009, there were 1570 bears in a wildlife refuge. In 2010, the population had increased to approximately 1884 bears. If this trend continues and the bear population is increasing exponentially, how many bears will there be in 2018?

\[ r = \frac{1884 - 1570}{1570} = 0.2 \]
\[ b = 1 + 0.2 = 1.2 \]
\[ 2018 - 2009 = 9 \]
\[ y = 1570 \left(1.2\right)^9 \approx 8101 \text{ bears} \]

14. The value of a piece of equipment has a decay factor of 0.80 per year. After 5 years, the equipment is worth $98,304. What was the original value of the equipment?

\[ A = \text{original value of equipment} \]
\[ 98,304 = A \left(0.80\right)^5 \Rightarrow 98,304 = \frac{A (0.80)^5}{(0.80)^5} \]
\[ A = 300,000 \]

15. Your friend drops a rubber ball from 4 ft. You notice that its rebound is 32.5 in. on the first bounce and 22 in. on the second bounce.

a. What exponential function would be a good model for the height of the ball?

\[ r = \frac{22}{32.5} = 0.677 \]
\[ b = 1 + (-0.323) \]
\[ 4 \text{ ft} = 48 \text{ in.} \text{, so } y = 48 \left(0.677\right)^x \]

b. How high will the ball bounce on the fourth bounce?

\[ y = 48 \left(0.677\right)^4 \approx 10.08 \text{ in.} \]

16. An investment of $75,000 increases at a rate of 12.5% per year. What is the value of the investment after 30 years?

\[ A = P \left(1 + r\right)^t \]
\[ A = 75,000 \left(1 + 0.125\right)^{30} \approx \$2,568,247.87 \]

17. A new truck that sells for $29,000 depreciates 12% each year. What is the value of the truck after 7 years?

\[ A = P \left(1 + r\right)^t \]
\[ = 29,000 \left(1 - 0.12\right)^7 \]
\[ = 29,000 \left(0.88\right)^7 \approx \$11,851.59 \]

18. The population of an endangered bird is decreasing at a rate of 0.75% per year. There are currently about 200,000 of these birds. What exponential function would be a good model for the population of these endangered birds and how many birds will there be in 100 years?

\[ y = 200,000 \left(1 - 0.0075\right)^x \]
\[ y = 200,000 \left(0.9925\right)^x \text{ model} \]
\[ y = 200,000 \left(0.9925\right)^{100} \]
\[ \approx 94,207 \text{ birds} \]
What are the definitions of a compression, reflection, and translation? Compression: shrink
translation: vertical or horizontal shift/slide; reflection: "flip"

Which transformation(s) can be eliminated? Why?

Reflection - the orientation of the function has not changed.

Could the transformation be a compression? A translation? Explain.

The transformation could be either a compression or a translation. You can divide
You can apply the four types of transformations - stretches, compressions, reflections, and translations - to exponential functions.

The graphs of \( y = 2^x \) (in red) and \( y = 3 \cdot 2^x \) (in blue) are shown. Each y-value of \( y = 3 \cdot 2^x \)
is three times (stretch factor of 3) the corresponding y-value of the parent function \( y = 2^x \).
EXAMPLE 1 – Graphing $y = ab^x$
How does the graph of $y = -\frac{1}{5} \cdot 4^x$ compare to the graph of the parent function?

It compresses the parent graph $y = 4^x$ by a factor of $\frac{1}{5}$, and reflects the graph in the x-axis.

TRY IT –
How does the graph of $y = -0.5 \cdot 5^x$ compare to the graph of the parent function?

It reflects in the x-axis and compresses by a factor of 0.5.

A horizontal shift $y = a(b^{x-h})$ is the same as a vertical stretch or compression $y = (ab^{-h})b^x$.
A vertical shift $y = ab^x + k$ also shifts the horizontal asymptote from $y = 0$ to $y = k$.

EXAMPLE 2 – Translating the Parent Function $y = b^x$
How does the graph of $y = 3(x+1)$ compare to the graph of the parent function?

It shifts the parent function $y = 3^x$ one unit to the left. The y-intercept becomes (0,3).

How does the graph of $y = 20(.5)^x + 10$ compare to the graph of the parent function?

Stretch the graph of $y = (.5)^x$ by a factor of 20 and translate the graph of $y = 20(.5)^x$ up 10 units. It also translates the asymptote, the y-intercept, and the range 10 units up. The asymptote becomes $y = 10$, the y-intercept becomes (0,30) and the range $(10, \infty)$ becomes $y > 10$. The domain is unchanged.

TRY IT –
How does the graph of $y = 4(x+2)$ compare to the graph of the parent function?

It shifts the parent function $y = 4^x$ 2 units to the left. The y-intercept becomes (0,16).
### Concept Summary: Families of Exponential Functions

- **Parent Function**
  \[ y = b^x \]
- **Stretch** \((|a| > 1)\)
- **Compression (Shrink)** \((0 < |a| < 1)\)
- **Reflection** \((a < 0)\) in x-axis
- **Translations** (horizontal by \(h\); vertical by \(k\))
  \[ y = b^{(x-h)} + k \]
- **All transformations combined**
  \[ y = ab^{(x-h)} + k \]

### Example 3 – Using an Exponential Model

Some insects reproduce exponentially. The chart shows the population of roaches in a colony at 36-day intervals. On what day will the colony reach 50,000,000 roaches?

\[
y = 45.86 \times (1.09)^x \]

during the 161st day

### Try It –

The best temperature to brew coffee is between 195°F and 205°F. Coffee is cool enough to drink at 185°F. The table shows temperature readings from a sample of coffee. How long does it take for the coffee to reach a temperature of 185°F?

\[
y = 196.66 \times (0.9776)^x \approx 2\frac{1}{2} \text{ min} - 3 \text{ min} \]
Up to this point you have worked with rational bases. However, exponential functions can have irrational bases as well. One important irrational base is the number $e$. The graph $y = (1 + \frac{1}{x})^x$ has an asymptote $y = e$ or $y \approx 2.71828$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = (1 + \frac{1}{x})^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y = 2$</td>
</tr>
<tr>
<td>10</td>
<td>$y \approx 2.594$</td>
</tr>
<tr>
<td>100</td>
<td>$y \approx 2.70$</td>
</tr>
<tr>
<td>1000</td>
<td>$y \approx 2.717$</td>
</tr>
</tbody>
</table>

As $x$ approaches infinity the graph approaches the value of $e$.

**Natural base exponential functions** are exponential functions with base $e$. These functions are useful for describing continuous growth or decay. Exponential functions with base $e$ have the same properties as other exponential functions.

**EXAMPLE 4 – Evaluating $e^x$**
What is the value of $2e^6$?

$\approx 806.86$

**TRY IT**
What is the value of $e^8$?

$\approx 2980.96$

We previously learned the formula for interest compounded annually. The formula for continuously compounded interest uses the number $e$.

### Key Concept: Continuously Compounded Interest

The formula for continuously compounded interest uses the number $e$:

$$A(t) = P \cdot e^{rt}$$

- $A(t)$: amount in account at time $t$
- $P$: principal
- $r$: interest rate (annual)
- $t$: time in years
EXAMPLE 5 – Continuously Compounded Interest
You have $1500 in a bank account that pays 4.5% annual interest compounded continuously. How much will you have in the account after 15 years? Round the answer to the nearest dollar.

\[ A = Pe^{rt} \]
\[ A = 1500e^{0.045 \times 15} \]
\[ A = \$2946 \]

TRY IT –
Suppose you won a contest at the start of 9th grade that deposited $3000 in an account that pays 5% annual interest compounded continuously. You start college 4 years later, and spend 4 years in college. About how much will be in the account after 4 years of college?

\[ A = Pe^{rt} \]
\[ A = 3000e^{0.05 \times 8} \]
\[ A = \$4475 \]

7-2 Lesson Quiz

1. How does the graph of \( y = 2 \cdot 2^x \) compare to the graph of the parent? \( \rightarrow y = 2 \cdot 2^x \) stretches the parent function \( y = 2^x \) by a factor of 2.

2. How does the graph of \( y = 4(x-6) \) compare to the graph of the parent function? \( y = 4(x-6) \) shifts the parent graph \( y = 4^x \) six units to the right and the \( y \)-intercept is now \((0, \frac{1}{4})\).

3. Do you UNDERSTAND? A pot of water is heated to 200°F. The table shows typical temperature readings for the pot. The room temperature is 70°F. How long will it take the water to cool to 150°F? \( y = 191.998 (0.9722)^x \rightarrow \approx 8 \text{ min} \)

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Temp (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>164</td>
</tr>
<tr>
<td>10</td>
<td>140</td>
</tr>
<tr>
<td>15</td>
<td>124</td>
</tr>
<tr>
<td>20</td>
<td>108</td>
</tr>
<tr>
<td>25</td>
<td>98</td>
</tr>
</tbody>
</table>

4. Your parents give you $10,000. You place it in an account that pays 6.1% annual interest compounded continuously. How much will you have in 20 years? Round the answer to the nearest dollar.

\[ A = Pe^{rt} \]
\[ A = 10,000e^{0.061 \times 20} \]
\[ A = \$33,871 \]
Graph each function.

1. \( y = 5(0.12)^x \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.167</td>
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<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>2</td>
<td>0.072</td>
</tr>
</tbody>
</table>

2. \( y = -0.1(5)^x \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-0.02</td>
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<td>-1</td>
</tr>
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<td>1</td>
<td>-5</td>
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<tr>
<td>2</td>
<td>-2.5</td>
</tr>
<tr>
<td>3</td>
<td>-12.5</td>
</tr>
</tbody>
</table>

3. \( y = -5 \left( \frac{1}{3} \right)^x \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
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<td>-1.15</td>
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<td>0</td>
<td>-1.5</td>
</tr>
<tr>
<td>1</td>
<td>-1.467</td>
</tr>
<tr>
<td>2</td>
<td>-1.559</td>
</tr>
</tbody>
</table>

Use the graph of \( y = e^x \) to evaluate each expression to four decimal places.

4. \( e^2 \approx 7.3891 \)

5. \( e^{-2.5} \approx 0.0821 \)

6. \( e^{1/3} \approx 1.3956 \)
Graph each function as a transformation of its parent function.

7. \( y = -(2)^{x+1} \)

8. \( y = -0.1(5)^{-x} \)

9. \( y = 2^x + 1 \)

Find the amount in a continuously compounded account for the given conditions.

10. principal: $20,000
    annual interest rate: 3.75%
    time: 2 yr

\[
A = Pe^{rt}
\]

\[
A = 20,000e^{(0.0375 \times 2)}
\]

\[
A = \$21,567.68
\]
11. **Error Analysis** A student says that the graph of \( f(x) = 2^{x+3} + 4 \) is a shift of 3 units up and 4 units to the right of the parent function. Describe and correct the student’s error.

The student reversed the horizontal and vertical translations of \( h \) and \( k \). The graph shifts the parent function left 3 units and 4 units up.

12. The isotope Sr-85 is used in bone scans. It has a half-life of 64.9 days. Write the exponential decay function for an 8-mg sample. Find the amount remaining after 100 days.

\[
y = 8 \left( 0.5 \right)^{\frac{x}{64.9}} \quad x = \text{days}
\]

\[
y = 8 \left( 0.5 \right)^{100/64.9}
\approx 2.75 \text{ mg}
\]

13. Suppose you invest $2000 at an annual interest of 5.5% compounded continuously.

a. How much will you have in the account in 10 years?

\[
A = Pe^{rt}
\]

\[
A = 2000 e^{(0.055 \times 10)} \approx 3466.50
\]

b. How long will it take for the account to reach $5000?

using a graphing calculator \( \approx 17 \text{ yrs} \)

The parent function for each graph below is of the form \( y = ab^x \). Write the parent function. Then write a function for the translation indicated.

14. Translation: left 3 units, up 1 unit

\[
y = 2^x \quad \text{is the parent function}
\]

\[
y = 2^{x+3} + 1
\]

15. Translation: right 3 units, up 1 units

The parent function is \( y = -4 \left( \frac{1}{2} \right)^x \).

Evaluating the pts \((0, -4); (1, -2)\) and \((2, -1)\) the function for the translation indicated is

\[
y = -4 \left( \frac{1}{2} \right)^{x-3} + 1
\]
A Closer Look at Compounding

The formula for finding the amount of money accumulated in an account is

\[ A = P \left( 1 + \frac{r}{n} \right)^{nt} \]

The variable \( P \) represents the principal, or amount initially invested.
The variable \( r \) represents the interest rate as a decimal.
The variable \( n \) represents the number of times per year the interest is compounded.
The variable \( t \) represents the time, or number of years for which the money is invested.

1. $750 is invested at 11\% compounded quarterly. How much is in the account after 10 yr? $2,219.9

\[ A = 750 \left( 1 + \frac{0.11}{4} \right)^{4(10)} \rightarrow 750 (1.0275)^{40} \]

2. Write the new formula for \( P = 51, r = 1.0, \) and \( t = 1 \) yr. \( A = 1 \left( 1 + \frac{r}{n} \right)^n \)

3. Remember that \( n \) is the number of times the interest is compounded. What happens as \( n \) grows? In other words, what is the effect of compounding more often? Fill in the following table. Round answers to eight decimal places.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \left( 1 + \frac{1}{n} \right)^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>2.54374246</td>
</tr>
<tr>
<td>100</td>
<td>2.70481383</td>
</tr>
<tr>
<td>1,000</td>
<td>2.71692393</td>
</tr>
<tr>
<td>10,000</td>
<td>2.71814593</td>
</tr>
<tr>
<td>100,000</td>
<td>2.71826824</td>
</tr>
<tr>
<td>1,000,000</td>
<td>2.71828047</td>
</tr>
<tr>
<td>10,000,000</td>
<td>2.71828181</td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>2.71828183</td>
</tr>
</tbody>
</table>

4. The table suggests that as \( n \) increases, the value of \( \left( 1 + \frac{1}{n} \right)^n \) gets closer to \( 2.71828183 \). If the value of \( n \) is increased further, the decimal approximation in the table will get very close to the value of a number known as \( e \). This number is used in many growth and decay applications. \( 2.71828183 \)

5. As \( n \) grows, you get closer to compounding continuously. This is why the formula used for compounding continuously is \( A = Pe^{rt} \). Rework Exercise 1 assuming that compounding is continuous. \( A = 750 e^{0.11(10)} \)

\[ = 750 e^{1.1} \]
KEY TERMS

Logarithm  Logarithmic Function  Common Logarithm  Logarithmic Scale

7-3 Solve It!

The chart shows the different ways you can write 4 and 16 in the form $a^b$, in which $a$ and $b$ are positive integers and $a \neq 1$.

What is the smallest number you can write in this $a^b$ form in four different ways?

In five different ways? In seven different ways? Explain how you found your answers.

What operation can you do to 4 to get 16?

- You can multiply 4 by 4
- You can square 4

What is the pattern of smallest numbers?

Each # must be a power of 2 whose exponent can be factored in the required # of ways.

What must be the smallest base of each of the smallest numbers? Explain.

2

Many even numbers can be written as power functions with base 2. In this lesson we will find ways to express all numbers as powers of a common base.

ESSENTIAL UNDERSTANDING – The exponential function $y = b^x$ is one-to-one, so its inverse $x = b^y$ is a function. To express "$y$ as a function of $x^n$" for the inverse, write $y = \log_b x$.

**Take Note**

**Key Concept**

A logarithm base $b$ of a positive number $x$ satisfies the following definition.

For $b > 0$, $b \neq 1$, \[ \log_b x = y \text{ if and only if } b^y = x. \]

You can read $\log_b x$ as "log base $b$ of $x$." In other words, the logarithm $y$ is the exponent to which $b$ must be raised to get $x$. 

\[
\begin{align*}
64 &= 2^6 \\
(2^3)^2 &= (2^3)^{2} \\
(2^3)^1 &= (2^3)^{1} \\
4096 &= 2^{12} \\
(2^3)^8 &= (2^3)^{8} \\
(2^4)^{3} &= (2^4)^{3} \\
16, 777, 216 &= 2^{12} \\
(2^4)^{3} &= (2^4)^{3} \\
(2^3)^8 &= (2^3)^{8} \\
(2^3)^{2} &= (2^3)^{2} \\
\end{align*}
\]
Expressed another way, the exponent \( y \) in the expression \( b^y \) is the logarithm in the equation \( \log_b x = y \). The base \( b \) in \( b^y \) and the base \( b \) in \( \log_b x \) are the same. In both, \( b \neq 1 \) and \( b > 0 \). And since \( b \neq 1 \) and \( b > 0 \), it follows that \( b^y > 0 \). Since \( b^y = x \) then \( x > 0 \), so \( \log_b x \) is defined only for \( x > 0 \).

Because \( y = b^x \) and \( \log_b x = y \) are inverse functions, their compositions map a number \( a \) to itself. In other word, \( b^{\log_b a} = a \) for \( a > 0 \) and \( \log_b b = a \) for all \( a \). We can use the definition of a logarithm to write exponential equations into logarithmic form.

**EXAMPLE 1** – Writing Exponential Equations in Logarithmic Form

What is the logarithmic form of each equation?

\[ a. \ 8^0 = 1 \]
\[ \log_8 1 = 0 \]

\[ b. \ 4^3 = 64 \]
\[ \log_4 64 = 3 \]

**TRY IT**

What is the logarithmic form of \( 36 = 6^2 \)?

\[ \log_6 36 = 2 \]

You can use the exponential form to help evaluate logarithms.

**EXAMPLE 2**

What is the value of \( \log_{16} 64 = \frac{3}{2} \)?

\[ \log_{16} 64 = y \rightarrow 16^y = 64 \]
\[ (4^2)^y = 4^3 \]

**TRY IT**

What is the value of \( \log_5 125 = 3 \)?

\[ \frac{3}{2} \rightarrow y = \frac{3}{2} \]

\[ 5^y = 125 \]
\[ 5^y = 5^3 \rightarrow y = 3 \]

A **common logarithm** is a logarithm with the base 10. You can write a common logarithm \( \log_{10} x \) simply as \( \log x \), without showing the 10.
Many measurements of physical phenomena have such a wide range of values that the reported measurements are logarithms (exponents) of the values, not the values themselves. When you use the logarithm of a quantity instead of the quantity, you are using a logarithmic scale. The Richter scale is a logarithmic scale. It gives logarithmic measurements of earthquake magnitude.

**The Richter Scale**

<table>
<thead>
<tr>
<th>magnitude:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>energy released:</td>
<td>E</td>
<td>E·10</td>
<td>E·10²</td>
<td>E·10³</td>
<td>E·10⁴</td>
<td>E·10⁵</td>
<td>E·10⁶</td>
<td>E·10⁷</td>
<td>E·10⁸</td>
<td>E·10⁹</td>
</tr>
</tbody>
</table>

**EXAMPLE 3 – Using a Logarithmic Scale**

In 1995, an earthquake in Mexico registered 8.0 on the Richter scale. In 2001, an earthquake of magnitude 6.8 shook Washington State. How many times more intense was the 1995 earthquake than the 2001 earthquake?

Use the formula \( \log \frac{I_1}{I_2} = M_1 - M_2 \) to compare the intensity levels of earthquakes, where \( I \) is the intensity level and \( M \) is the magnitude on the Richter scale.

\[
\log \frac{I_1}{I_2} = 8.0 - 6.8 \Rightarrow \log \frac{I_1}{I_2} = 1.2
\]

\[
\Rightarrow 10^{1.2} = \frac{I_1}{I_2} \approx 15.85 \text{ times more intense}
\]

**TRY IT –**

The loudness of a sound in decibels, dB, is defined as \( 10 \log \frac{I}{10^{-10}} \), where \( I \) is the intensity of the sound. How loud is a whisper with an intensity of \( 10^{-10} \)?

\[
10 \log \frac{10^{-10}}{10^{-12}} \Rightarrow 10 \log 10^2
\]

\[
\log 10^2 \Rightarrow 10^y = 10^2 \\
y = 2
\]

\[
10(2) = 20 \text{ dB}
\]
A **logarithmic function** is the inverse of an exponential function. The graph shows $y=10^x$ and its inverse $y=\log x$. Note that $(0, 1)$ and $(1, 10)$ are on the graph $y=10^x$, and that $(1, 0)$ and $(10, 1)$ are on the graph of $y=\log x$.

Recall that the graphs of inverse functions are reflections of each other across the line $y=x$. You can graph $y=\log_b x$ as the inverse of $y=b^x$.

**EXAMPLE 4 – Graphing a Logarithmic Function**
What is the graph of $y=\log_6 x$? Describe the domain and range and identify the $y$-intercept and the asymptote.

$y=\log_6 x$

- **Domain (D)**: $(0, \infty)$
- **Range (R)**: $(-\infty, \infty)$
- **Asymptote**: $x=0$
- **$y$-axis**: no $y$-int.

**TRY IT –**
What is the graph of $y=\log_4 x$? Describe the domain and range and identify the $y$-intercept and the asymptote.

$y=\log_4 x$

- **Domain (D)**: $(0, \infty)$
- **Range (R)**: $(-\infty, \infty)$
- **Asymptote**: $y$-axis
- **$x=0$**: no $y$-int.
The function \( y = \log_b x \) is the parent for a function family. You can graph 
\( y = \log_b(x - h) + k \) by translating the graph of the parent function \( y = \log_b x \), horizontally by \( h \) units and vertically by \( k \) units. The \( a \) in \( y = \log_b ax \) indicates a stretch, a compression, and possibly a reflection.

**Concept Summary**

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parent functions: ( y = \log_b x, b &gt; 0, b \neq 1 )</td>
<td></td>
</tr>
<tr>
<td>Stretch ( (</td>
<td>a</td>
</tr>
<tr>
<td>Compression (Shrink) ( (0 &lt;</td>
<td>a</td>
</tr>
<tr>
<td>Reflection ( (a &lt; 0) ) in ( x )-axis</td>
<td></td>
</tr>
<tr>
<td>Translations (horizontal by ( h ); vertical by ( k ))</td>
<td></td>
</tr>
<tr>
<td>All transformations together</td>
<td></td>
</tr>
</tbody>
</table>

**EXAMPLE 5** – Translating \( y = \log_b x \)

How does the graph of \( y = \left(\frac{3}{4}\right) \log x - 2 \) compare to the graph of the parent function?

- **Compression by a factor of \( \frac{3}{4} \) and a
  - Transition vertically 2 units up.
  - Domain, range and asymptote remains the same.**

**TRY IT** –

How does the graph of \( y = \log_b(x - 3) + 4 \) compare to the graph of the parent function?

- **Translation of the graph 3 units to the right and 4 units up.** The asymptote changes from \( x = 0 \) to \( x = 3 \). The domain changes from \( (0, \infty) \) to \( (3, \infty) \). The range does **NOT** change.
The pH Scale

The acidity or alkalinity of a substance is measured on another logarithmic scale, called the pH scale.

\[ pH = -\log(H^+) \]

**pH Scale**

<table>
<thead>
<tr>
<th>pH</th>
<th>Concentration of H(^+) (moles/liter)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>(10^{-14})</td>
</tr>
<tr>
<td>13</td>
<td>(10^{-13})</td>
</tr>
<tr>
<td>12</td>
<td>(10^{-12})</td>
</tr>
<tr>
<td>11</td>
<td>(10^{-11})</td>
</tr>
<tr>
<td>10</td>
<td>(10^{-10})</td>
</tr>
<tr>
<td>9</td>
<td>(10^{-9})</td>
</tr>
<tr>
<td>8</td>
<td>(10^{-8})</td>
</tr>
<tr>
<td>7</td>
<td>(10^{-7}) H₂O</td>
</tr>
<tr>
<td>6</td>
<td>(10^{-6})</td>
</tr>
<tr>
<td>5</td>
<td>(10^{-5})</td>
</tr>
<tr>
<td>4</td>
<td>(10^{-4})</td>
</tr>
<tr>
<td>3</td>
<td>(10^{-3})</td>
</tr>
<tr>
<td>2</td>
<td>(10^{-2})</td>
</tr>
<tr>
<td>1</td>
<td>(10^{-1})</td>
</tr>
<tr>
<td>0</td>
<td>(10^{0})</td>
</tr>
</tbody>
</table>

What is the pH of a solution that has a concentration of hydrogen ions of \(10^{-9}\)?

\[ \text{pH of 9} \]

What is the concentration of hydrogen ions in a solution with a pH of 5?

\(10^{-5}\)
Decibel Scale

<table>
<thead>
<tr>
<th>Watts/Square Meter</th>
<th>Decibels</th>
</tr>
</thead>
<tbody>
<tr>
<td>10⁻¹²</td>
<td>0</td>
</tr>
<tr>
<td>10⁻¹¹</td>
<td>10</td>
</tr>
<tr>
<td>10⁻¹⁰</td>
<td>20</td>
</tr>
<tr>
<td>10⁻⁹</td>
<td>30</td>
</tr>
<tr>
<td>10⁻⁸</td>
<td>40</td>
</tr>
<tr>
<td>10⁻⁷</td>
<td>50</td>
</tr>
<tr>
<td>10⁻⁶</td>
<td>60</td>
</tr>
<tr>
<td>10⁻⁵</td>
<td>70</td>
</tr>
<tr>
<td>10⁻⁴</td>
<td>80</td>
</tr>
<tr>
<td>10⁻³</td>
<td>90</td>
</tr>
<tr>
<td>10⁻²</td>
<td>100</td>
</tr>
<tr>
<td>10⁻¹</td>
<td>110</td>
</tr>
<tr>
<td>10⁰</td>
<td>120</td>
</tr>
<tr>
<td>10¹</td>
<td>130</td>
</tr>
<tr>
<td>10²</td>
<td>140</td>
</tr>
<tr>
<td>jet plane (30 m away)</td>
<td>140</td>
</tr>
<tr>
<td>pain level</td>
<td>130</td>
</tr>
<tr>
<td>amplified rock music (2 m)</td>
<td>120</td>
</tr>
<tr>
<td>noisy kitchen</td>
<td>100</td>
</tr>
<tr>
<td>heavy traffic</td>
<td>90</td>
</tr>
<tr>
<td>normal conversation</td>
<td>60</td>
</tr>
<tr>
<td>average home</td>
<td>50</td>
</tr>
<tr>
<td>soft whisper</td>
<td>30</td>
</tr>
<tr>
<td>barely audible</td>
<td>0</td>
</tr>
</tbody>
</table>

**7-3 Lesson Quiz**

1. What is the logarithmic form of \(144 = 12^2\)?
   \[ \log_{12} 144 = 2 \]

2. What is the value of \(\log_9 27\)?
   \[ 9^x = 27 \rightarrow (3^2)^x = 3^3 \rightarrow 2x - 2 \rightarrow x = \frac{3}{2} \]
   \[ \log_9 27 = \frac{3}{2} \]

3. Do you UNDERSTAND? The pH of a substance equals \(-\log[H^+]\), where \([H^+]\) is the concentration of hydrogen ions. \([H^+]\) for tomato juice is \(10^{-4}\). What is the pH of tomato juice?
   \[ -\log[H^+] = \text{pH} \rightarrow -\log 10^{-4} = y \rightarrow 10^x = 10^{-4} \]
   \[ \text{so } x = -4 \]
   \[ \log 10^{-4} = -4 \text{, so } -\log 10^{-4} = 4 \]

4. What is the graph of \(y = \log_5 x\)? Describe the domain and range, and identify the y-intercept and the asymptote.

5. How does the graph of \(y = 2\log(x + 5)\) compare to the graph of the parent function?
   \[ y = \log_5 x \]
   \[ 5^y = x \]

4. The graph is stretched by a factor of 2 and translated to the left 5 units.
   - Domain: \((0, \infty)\)
   - Range: \(R\)
   - Y-intercept: None
   - Asymptote: \(x = 0\) on y-axis

5. The domain changes to \((5, \infty)\), range remains the same; y-intercept \((0, 1.37)\), asymptote \(x = -5\)
Write each equation in logarithmic form.

1. \(9^2 = 81 \rightarrow \log_9 81 = 2\)
2. \(8^3 = 512 \rightarrow \log_8 512 = 3\)
3. \(2^5 = 512 \rightarrow \log_2 512 = 9\)
4. \(5^4 = 625 \rightarrow \log_5 625 = 4\)

Evaluate each logarithm.

5. \(\log_4 128 = 7\)
   \[2^x = 128 \rightarrow 2^x = 2^7, \text{ so } x = 7\]
6. \(\log_9 27 = \frac{3}{2}\)
   \[3^x = 27 \rightarrow (3^2)^x = 3^3, \text{ so } \frac{2x}{2} = \frac{3}{2} \rightarrow x = \frac{3}{2}\]
7. \(\log_{\frac{1}{9}} \frac{1}{9} = 2\)
   \[\left(\frac{1}{3}\right)^x = \frac{1}{9} \rightarrow \left(3^{-1}\right)^x = 3^{-2}, \text{ so } -\frac{x}{-1} = -\frac{2}{-1} \rightarrow x = 2\]
8. \(\log_7 1 = 0\)
   \[7^0 = 1, \text{ so } x = 0\]

In 2004, an earthquake of magnitude 7.0 shook Papua, Indonesia. Compare the intensity level of that earthquake to the intensity level of each earthquake below.

9. magnitude 6.1 in Costa Rica, in 2009
   \[\log \frac{I_1}{I_2} = 7.0 - 6.1 \rightarrow \log \frac{I_1}{I_2} = 0.9 \rightarrow 10^{0.9} = \frac{I_1}{I_2} \rightarrow \text{so the Papua earthquake was 8 times as strong as the Costa Rica.}\]

10. magnitude 7.8 in the Fiji Islands, in 2007
    \[\log \frac{I_1}{I_2} = 7.8 - 7.0 \rightarrow \log \frac{I_1}{I_2} = 0.8 \rightarrow 10^{0.8} = \frac{I_1}{I_2} \rightarrow \text{the Fiji Islands' earthquake was about 6 times as strong as the Papua earthquake.}\]

Describe how the graph of each function compares with the graph of the parent function, \(y = \log_3 x\).

11. \(y = \log_3 (x - 8)\) translates 8 units to the right, domain \((8, \infty)\), range is the same, asymptote becomes \(x = 8\), no shift.

12. \(y = \log_3 (x - 4) + 1\) translated 4 units to the right and one unit up, domain \((4, \infty)\), range remains the same, asymptote becomes \(x = 4\), no shift.
Graph each logarithmic function.

13. \( y = \log x \)  
   \[ (10^x = y) \rightarrow y = 10^x \quad y = \log x \]

14. \( y = \log_6 x \)  
   \[ (6^x = y) \rightarrow y = 6^x \quad y = \log_6 x \]

Write each equation in exponential form.

15. \( \log_7 1 = 0 \)
   \[ 7^0 = 1 \]

16. \( \log 10 = 1 \)
   \[ 10^1 = 10 \]

17. \( \log_5 6561 = 8 \)
   \[ 3^8 = 6561 \]

18. \( \log_{17} 289 = 2 \)
   \[ 17^2 = 289 \]

19. \( \log_{12} \frac{1}{144} = -2 \)
   \[ 12^{-2} = \frac{1}{144} \]

20. \( \log_{64} \frac{1}{64} = -2 \)
   \[ 8^{-2} = \frac{1}{64} \]

21. A single-celled bacterium divides every hour. The number \( N \) of bacteria after \( t \) hours is given by the formula \( \log_2 N = t \). After how many hours will there be 32 bacteria?

   \[ \log_2 32 = t \rightarrow 2^t = 32 \rightarrow 2^t = 2^5 \]
   \[ t = 5 \text{ hrs} \]

For each pH given, find the concentration of hydrogen ions \([H^+]\).
Use the formula \( pH = - \log[H^+] \).

22. \( pH = 7.2 \)
   \[ 10^{-7.2} \rightarrow 6.31 \times 10^{-8} \]

23. \( pH = 8.2 \)
   \[ 10^{-8.2} \rightarrow 6.31 \times 10^{-9} \]

24. \( pH = 5.6 \)
   \[ 10^{-5.6} \rightarrow 2.51 \times 10^{-6} \]

25. \( pH = 7.0 \)
   \[ 10^{-7} \rightarrow 1.0 \times 10^{-7} \]
Find the inverse of each function.

26. \( y = \log_2 x \) \implies 2^y = x
   \[ 2^y = y \text{ is the inverse} \]

27. \( y = \log_{100} x \)
   \[ 100^y = x \]
   \[ 100^y = y \text{ the inverse} \]

28. \( y = \log_4 (4x) \)
   \( 2^4 = 4x \) \implies \( \frac{2^x}{4} = y \)
   \[ \frac{2^x}{2^2} = y \implies y = 2^{x-2} \text{ the inverse} \]

Find the domain and range of each function.

29. \( y = \log_3 x - 2 \)
   \[ y + 2 = \log_3 x \implies 3^{y+2} = x \]
   The log has the domain and range opposite the inverse.
   \[ y = 3^{x+2} \text{ is inverse} \]
   \[ D: (-\infty, \infty) \quad R: (0, \infty) \]

30. \( y = \log (x + 1) \)
   \[ 10^y = (x + 1) \]
   \[ 10^x = y + 1 \]
   \[ -1 \]
   \[ 10^x - 1 = y \text{ is inverse} \]
   \[ D: (-\infty, \infty) \]
   \[ R: (-1, \infty) \]
   So the domain and range of \( y = \log (x+1) \) is
   \[ D: (-1, \infty) \]
   \[ R: (-\infty, \infty) \]
7-3

Enrichment

Log Jams

The logarithm is a tool originally developed and used to aid in calculations, yet this viewpoint of logarithms is not the only one of interest. Logarithms are also useful when thought of as real-valued functions, or as inverse functions of the corresponding exponential functions. The idea of a logarithm as an inverse function of an exponential function means that \( \log_a x \) is a question to be answered. For example, you can read the expression \( \log_2 8 \) as "what exponent on base 2 gives 8?" The answer is 3, because \( 2^3 = 8 \).

Thinking of a logarithm as an exponent helps to order some logarithms without evaluating them. For example, the logarithms \( \log_2 8, \ log_3 7, \) and \( \log_5 6 \) are in descending order since the exponent needed on base 7 that gives 8 would be greater than 1, and 1 is in turn greater than the exponent needed on base 7 that gives 6.

You can also compose logarithms as you would compose other functions, where their domain and ranges agree. Thus, you evaluate \( \log_4 (\log_3 25) \) by evaluating \( \log_3 25 = 2 \), then evaluating \( \log_2 2 = \frac{1}{2} \).

Rewrite each equation in exponential form to solve the equation.

1. Solve for \( x \): \( \log_4 81 = 4 \)
   \[
   (x^4)^{1/4} = (81)^{1/4} \Rightarrow x = 3
   \]
2. Solve for \( x \): \( \log_2 2 = 2 \)
   \[
   (x^2)^{1/2} = (2)^{1/2} \Rightarrow x = \sqrt{2}
   \]
3. Which is greater, \( \log_2 3 \) or \( \log_3 2 \)?
   \[
   2^x = 3 \Rightarrow 3^x = 2 , \ so \ \log_2 3 \ is \ greater
   \]
4. Solve for \( x \): \( \log_3 x = \log_3 3 \)
   \[
   \log_3 x = \log_3 3 \Rightarrow x = 3
   \]
5. Which is greater, \( \frac{1}{3} \) of \( \log_2 2 \) or \( \frac{1}{2} \) of \( \log_10 10 ? \)
   \[
   \frac{1}{3} \log_2 2 = \frac{1}{3} = 0.33 \frac{1}{2} \log_10 10 = 1
   \]
6. Solve for \( x \): \( \log_2 (\log_3 x) = 2 \)
   \[
   2^x = \log_3 x \Rightarrow 3^x = x \Rightarrow x = 10
   \]
7. Which is greater, \( \frac{1}{2} \log_2 (\log_8 8.5) \) or \( \frac{1}{2} \log_3 (\log_9 8.5) \)?
   \[
   \frac{1}{2} \log_2 (\log_8 8.5) = \log_2 10 \log_2 3 = \frac{1}{2} \log_2 10 \log_2 3 = 1.0
   \]
8. Which of the following are equal?
   \[
   \log_2 \left( \frac{1}{2} \right) = \log_2 1 - \log_2 2 = 0
   \]
9. \( \log_5 5x = 1 \)
10. \( \log_2 (2x^2 - 7) = 0 \)
   \[
   2^0 = 2x^2 - 7 \Rightarrow 1 = 2x^2 - 7
   \]
11. \( \log_3 1 = x \)
12. \( \log_2 x^2 = 2 \)
   \[
   4 = x^2 \Rightarrow x = \pm 2
   \]
13. \( \log_2 (x^2 + 1) = 2 \)
14. \( \log_3 x = x \)
   \[
   x = 0 \Rightarrow x = \pm 2
   \]
15. \( \log_3 1 = x \)
16. \( \log_5 x^2 = 2 \)
   \[
   4 = x^2 \Rightarrow x = \pm 2
   \]
17. \( \log_2 (x^2) = 2 \)
18. \( \log_3 (x + 1) = 0 \)
   \[
   4^0 = (x + 1) \Rightarrow x = 0
   \]
19. \( 1 + \log_2 (x - 1) = 1 \)
20. \( -1 + \log_3 x = -1 \)
   \[
   1 = x - 1 \Rightarrow x = 2
   \]

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Log Jams

The logarithm is a tool originally developed and used to aid in calculations, yet this viewpoint of logarithms is not the only one of interest. Logarithms are also useful when thought of as real-valued functions, or as inverse functions of the corresponding exponential functions. The idea of a logarithm as an inverse function of an exponential function means that \( \log_b x \) is a question to be answered. For example, you can read the expression \( \log_8 8 \) as "what exponent on base 2 gives 8?" The answer is 3, because \( 2^3 = 8 \).

Thinking of a logarithm as an exponent helps to order some logarithms without evaluating them. For example, the logarithms \( \log_8 8, \log_7 7, \) and \( \log_6 6 \) are in descending order since the exponent needed on base 7 that gives 8 would be greater than 1, and 1 is in turn greater than the exponent needed on base 7 that gives 6.

You can also compose logarithms as you would compose other functions, where their domain and ranges agree. Thus, you evaluate \( \log_2 (\log_2 25) \) by evaluating \( \log_2 25 = 2 \), then evaluating \( \log_2 2 = \frac{1}{2} \).

Rewrite each equation in exponential form to solve the equation.

1. Solve for \( x \): \( \log_8 81 = 4 \) → \( (8^4)^{\frac{1}{4}} = 8^1 \) → \( x = 3 \)
2. Solve for \( x \): \( \log_2 2 = 2 \) → \( 2^2 = 2 \) → \( x = \sqrt{2} \)
3. Which is greater, \( \log_3 3 \) or \( \log_8 2 \)?
   → \( 2^y = 3 \) or \( 3^y = 2 \)
4. Solve for \( x \): \( \log_6 2 = \log_3 3 \)
5. Which is greater, \( \frac{1}{3} \) of \( \log_4 2 \) or \( \frac{1}{2} \) of \( \log_10 10 \)?
6. Solve for \( x \): \( \log_2 (\log_2 x) = 2 \) → \( 2^2 = \log_2 x \) → \( 4 = \log_2 x \) → \( 2^4 = x \)
7. Which is greater, \( \frac{1}{3} \log_2 (\log_3 8.5) \) or \( \frac{1}{2} \log_3 (\log_3 8.5) \)?
8. Which of the following are equal?
   \( \log_2 \frac{1}{2} \) \quad \( \frac{\log_1}{\log_2} \) \quad \log_1 - \log_2

Rewrite in exponential form and solve for \( x \).

9. \( \log_5 x = 1 \) → \( 5^1 = x \) → \( x = 1 \)
10. \( \log_2 (2x^2 - 7) = 0 \) → \( 2^0 = 2x^2 - 7 \) → \( 2x^2 = 4 \) → \( x^2 = 2 \) → \( x = \pm \sqrt{2} \)
11. \( \log_7 x^2 \) → \( x^2 = 7 \) → \( x = \pm \sqrt{7} \)
12. \( \log_7 17 = x \) → \( 17^x = 17 \) → \( x = 1 \)
13. \( \log_3 1 \) → \( x = 0 \)
14. \( \log_3 3 \) → \( 3^1 = x \) → \( x = 1 \)
15. \( \log_3 3^2 = x \) → \( 3^2 = 3^x \) → \( x = 2 \)
16. \( \log_3 (x + 1) = 0 \) → \( 3^0 = x + 1 \) → \( x = -1 \)
17. \( 1 + \log_3 (x - 1) = 1 \) → \( x = 2 \)
18. \( -1 + \log_3 (x + 1) = 0 \) → \( 3^0 = x + 1 \) → \( x = 0 \)
19. \( 0 = (x - 1) \) → \( 1 = x - 1 \) → \( x = 2 \)
20. \( -1 + \log_3 (x - 1) = 0 \) → \( 3^1 = x - 1 \) → \( x = 2 \)

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EXAMPLE 1
What type of function models the data best — linear, logarithmic, or exponential?

1. Press \textbf{Stat} \textbf{enter} to enter the data list.
2. Use scatter plot.
3. Determine that it is \textit{exponential}.

EXAMPLE 2
What type of function models the data best — quadratic, logarithmic, or cubic?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.4</td>
</tr>
<tr>
<td>1</td>
<td>7.5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1.8</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
</tr>
<tr>
<td>5</td>
<td>3.9</td>
</tr>
</tbody>
</table>

\textbf{Exercises}

1. Which type of function models the data shown in the graphing calculator screen best — linear, quadratic, logarithmic, cubic, or exponential?

2. Which type of function models the data in the table best — linear, quadratic, logarithmic, cubic, or exponential?

3. \textbf{Reasoning} Could you use a different model for the data in Exercises 1 and 2? Explain.

\textbf{Yes—problem 1}
Could be linear with outliers.

\textbf{problem 2 could also be}
be linear or cubic. They cannot
be logarithmic due to negative
values.